Announcements

1) PIC Math! Math 390E course in Winter term – work on math problems provided by industry (Ford, etc.) – programming!

2) Exam 3 covers 14.7, 14.8 15.1 – 15.4

In general

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (g(x,y), h(x,y))$

Let

$$J_T(x,y) = \det\left(\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \end{bmatrix}\right)$$

Let R be a region in $\mathbb{R}^2$,

$f, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ continuous on R.

If T is one-to-one on R and $J_T \neq 0$ on R,

$$\boxed{\int_{T(R)} f(x,y)\,dA = \int_R f(T(x,y))\,|J_T(x,y)|\,dA}$$

__Example 1__ : $T(x,y) = (2x, 2y)$

Compute $J_T(x,y)$

$g(x,y) = 2x$

$h(x,y) = 2y$

$\frac{\partial g}{\partial x} = 2$, $\frac{\partial g}{\partial y} = 0$

$\frac{\partial h}{\partial x} = 0$, $\frac{\partial h}{\partial y} = 2$

$$J_T(x,y) = \det \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= 4$$

Example 2 Compute J_T if

$$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$

$$x(r,\theta) = r\cos(\theta)$$

$$y(r,\theta) = r\sin(\theta)$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta)$$

$$J_T(r,\theta) = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

$$= r \quad \checkmark$$

This where the "r" comes from in polar coordinate integration

Observation: (polar 1-1)

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

is not 1-1 when

$r \le 0$:

$$T(0,\theta) = (0,0)$$

for all values of θ.

Moreover, if $r>0$,

$$T(-r,\theta) = (-r\cos\theta, -r\sin\theta)$$

$$= (r\cos(\theta+\pi), r\sin(\theta+\pi))$$

$\boxed{\text{So we need } r>0}$

Finally, since sine and cosine are periodic of period 2π, we need $\boxed{0 \le \theta < 2\pi}$ for T to be one-to-one.

Triple Integrals

(Section 15.7)

Integration in $\mathbb{R}^3$!

Let $B = [a,b] \times [c,d] \times [e,f]$, a box in $\mathbb{R}^3$.

$$B = \{(x,y,z) \mid a \le x \le b, \; c \le y \le d, \; e \le z \le f\}.$$

Chop into pieces:

Chop $[a,b]$ into n pieces,

$[c,d]$ into m pieces

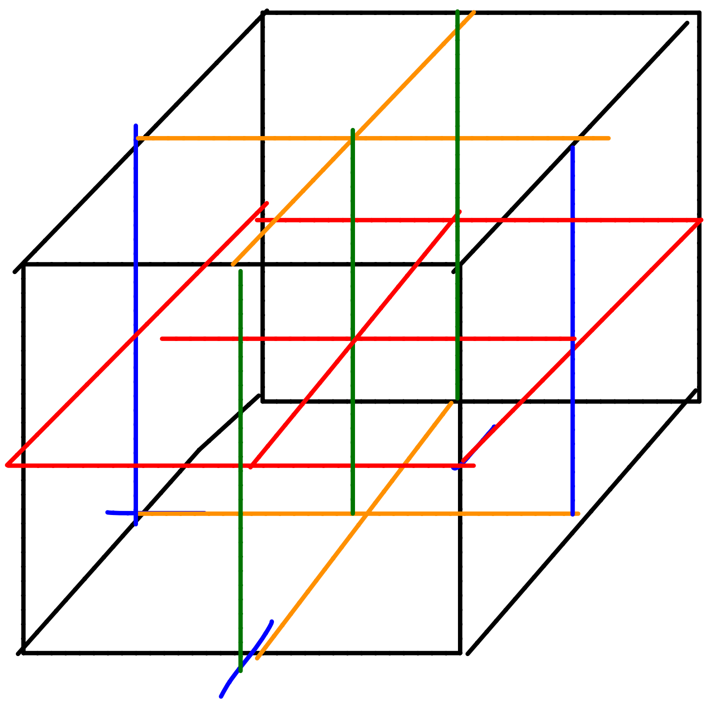
$[e,f]$ into ℓ pieces.

All pieces have the same

length $\frac{b-a}{n}$, $\frac{d-c}{m}$, $\frac{f-e}{\ell}$,

respectively

Picture



$n = m = \ell = 2$

8 interior boxes

The Definite Integral

Let $B = [a,b] \times [c,d] \times [e,f]$ be a box in $\mathbb{R}^3$. If $f = f(x,y,z)$ is continuous on B and real-valued, define

volume

$$\iiint_B f(x,y,z)\, dV$$ as

$$\iiint_B f(x,y,z)\, dV$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} \lim_{l \to \infty} \frac{b-a}{n} \frac{d-c}{m} \frac{f-e}{l} \cdot \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(x_{i,j,k}, y_{i,j,k}, z_{i,j,k}) \right)$$

where $(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$

is a point in

$$\left[a + \frac{(j-1)(b-a)}{n}, a + \frac{j(b-a)}{n}\right] \times \left[c + \frac{(i-1)(d-c)}{m}, c + \frac{i(d-c)}{m}\right] \times$$

$$\left[e + \frac{(k-1)(f-e)}{l}, e + \frac{k(f-c)}{l}\right]$$